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A Note on Janowski Functions With Respect To $(2j,k)$ - Symmetric Conjugate Points

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Abstract: *Using the concept of* $(2j,k)$ *- symmetrical conjugate points, the classes* $\mathcal{S}^*_{(2j,k)}(A,B), \mathcal{K}_{(2j,k)}(A,B)$ *are introduced. For functions belonging to these classes certain interesting properties are discussed.*

Keywords and Phrases: *Janowski functions, Convex Functions, Starlike Functions,* $(2j,k)$ *- Symmetric Conjugate Points.*

I. Introduction

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1.1)$$

Which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C}\ and\ |z| < 1\}$ and $\mathcal{S}$ denote subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathcal{U}$.

For f and g be analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$, if there exists an analytic function ω in $\mathcal{U}$, such that $|\omega(z)| < |z|$ and $f(z) = g(\omega(z))$, and we denote this by $f \prec g$. If g is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

We now introduce the concept of (j,k)-symmetrical functions which generalizes the concept of even, odd and *k*-symmetric functions. Consider,

$$e^{-i\alpha} f(e^{i\alpha} z) = z + \sum_{n=2}^{\infty} a_n e^{i(n-1)\alpha} z^n, \quad \alpha \in \mathbb{R} \qquad (1.2)$$

And $$\left[f(z^k)\right]^{\frac{1}{k}} = z + \frac{a_2}{k} z^{k+1} + \frac{1}{2k^2}\left[2ka_3 - (k-1)a_2^2\right] z^{2k-1} + \ldots. \qquad (1.3)$$

Where k is a positive integer. The transformation in (1.2) is a rotation of f because it rotates the unit disc in the z-plane through an angle α and rotates the image domain in the w-plane in the reverse detection through an angle of the same magnitude. In (1.3) the transformation $u = z^k$ maps $\mathcal{U}$ onto k copies of $\mathcal{U}$ and $f(z)$ carries this surface onto k copies of $f(\mathcal{U})$ joined by a suitable branch point at $w = 0$. It is intuitively clear that the k^{th} root merely unwinds the symmetry. Precisely, we have

Definition 1.1. Let k be a positive integer. A domain D is said to be k-fold symmetric if a rotation of D about the origin through an angle $\frac{2\pi}{k}$ carries D onto itself. A function f is said to be k-fold symmetric in $\mathcal{U}$ if for every z in $\mathcal{U}$

$$f\left(e^{2\pi i/k}z\right)=e^{2\pi i/k}f(z.)$$

The family of all k-fold symmetric function denoted by $\mathcal{S}^k$ and for $k=2$ we get the odd univalent function.

The notion of (j,k) - symmetric functions $(k=2,3....;\mathrm{j}=0,1,2,....,k-1)$ is a generalization of the notion of even, odd, k-symmetrical functions and generalize the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function.

The theory of (j,k) symmetrical functions has many interesting applications, for instance in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan uniqueness theorem for holomorphic mappings [12].

Definition 1.2. Let $\varepsilon=\left(e^{2\pi i/k}\right)$ and $j=0,1,2,...,k-1$ where $k\geq 2$ is a natural number. A function $f:\mathcal{U}\mapsto \mathbb{C}$ is called (j,k) - symmetrical if

$$f(\varepsilon z)=\varepsilon^j f(z),\ z\in\mathcal{U}.$$

The family of all (j,k) - symmetrical functions is denoted be $\mathcal{S}^{(j,k)}.\mathcal{S}^{(0,2)},\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(l,k)}$ are respectively the classes of even, odd and k-symmetric functions. We have the following decomposition theorem.

Theorem 1.3.[12] For every mapping $f:\mathcal{U}\mapsto\mathbb{C}$, there exists exactly the sequence of (j,k) - symmetrical functions $f_{j,k}$,

$$f(z)=\sum_{j=0}^{k-1}f_{j,k}(z)$$

Where

$$f_{j,k}(z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}f\left(\varepsilon^v z\right) \qquad (1.4)$$

$$\left(f\in A;k=1,2,...;\ j=0,1,2,...,k-1\right).$$

We denote by $\mathcal{S}^*,\mathcal{K},\mathcal{C},\mathcal{C}^*$ the familiar subclasses consisting of functions which, respectively, starlike, convex, close-to-convex and quasi-convex in $\mathcal{U}$.

Al-Amiri, Coman and Mocanu in [6] introduced and investigated a class of functions starlike with respect to 2*k*-symmetric conjugate points, which satisfy the following inequality

$$\mathrm{Re}\left(\frac{zf'(z)}{f_{2k}(z)}\right)>0,$$

Where $f_{2j,k}$ defined by

$$f_{2k}(z)=\frac{1}{2k}\sum_{v=0}^{k-1}\left[\varepsilon^{-v}f\left(\varepsilon^v z\right)+\varepsilon^v\overline{f\left(\varepsilon^v\overline{z}\right)}\right],\ \left(f\in\mathcal{A}\right).$$

Karthikeyan in [5] once introduced and investigated a class of function starlike with respect to $(2j,k)$ - symmetric conjugate points, and $\mathcal{S}^{*}_{(2j,k)}(\phi)$ denoted the class of function $f \in \mathcal{A}$ and satisfy the subordination condition,

$$\frac{zf'(z)}{f_{2j,k}(z)} \prec \phi(z),\ (z \in \mathcal{U}).$$

Where $f_{2j,k}$ is defined by (1.5) a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}_{(2j,k)}(\phi)$ if it satisfies the subordination condition,

$$\frac{(zf'(z))'}{f_{2j,k}(z)} \prec \phi(z),\ (z \in \mathcal{U}).$$

Where $\mathcal{P}$ class of functions with positive real part and

$$k = 1,2,3,....;\ \mathrm{j} = 0,1,2,...(k-1);\phi \in \mathcal{P}.$$

Where $f_{2j,k}$ is defined by

$$f_{2j,k}(z) = \frac{1}{2k}\sum_{v=0}^{k-1}\left[\varepsilon^{-vj} f\left(\varepsilon^{v} z\right) + \varepsilon^{vj}\,\overline{f\left(\varepsilon^{v}\overline{z}\right)}\right], (f \in \mathcal{A}). \qquad (1.5)$$

If v in an integer, then the following identities follow directly from (1.5)

$$f'_{2j,k}(z) = \frac{1}{2k}\sum_{v=0}^{k-1}\left[\varepsilon^{v-vj} f'\left(\varepsilon^{v} z\right) + \varepsilon^{vj-v}\,\overline{f'\left(\varepsilon^{v}\overline{z}\right)}\right], \qquad (1.6)$$

And

$$f_{2j,k}\left(\varepsilon^{v} z\right) = \varepsilon^{vj} f_{2j,k}(z),\ f_{2j,k}(z) = \overline{f_{2j,k}\left(\overline{z}\right)} \qquad (1.7)$$

$$f'_{2j,k}\left(\varepsilon^{v} z\right) = \varepsilon^{vj-v} f'_{2j,k}(z),\ f'_{2j,k}\left(\overline{z}\right) = \overline{f'_{2j,k}(z)} \qquad (1.8)$$

Definition 1.4. A function f in $\mathcal{A}$ is said to belong to the class $\mathcal{S}^{*}_{(2j,k)}(A,B),\ (-1 \le B < A \le 1)$ if

$$\frac{zf'(z)}{f_{2j,k}(z)} \prec \frac{1+Az}{1+Bz},\ z \in \mathcal{U}.$$

Where $f_{2j,k}(z)$ is defined by (1.5)

Definition 1.5. A function f in $\mathcal{A}$ is said to belong to the class $\mathcal{K}_{(2j,k)}(A,B), (-1 \le B < A \le 1)$ if

$$\frac{(zf'(z))'}{f'_{2j,k}(z)} \prec \frac{1+Az}{1+Bz},\ z \in \mathcal{U}.$$

Where $f_{2j,k}(z)$ is defined by (1.5)

In this paper we will prove that $\mathcal{S}^{*}_{(2j,k)}(A,B)$ and $\mathcal{K}_{(2j,k)}(A,B)$ are subclasses of the class of close to convex functions and the class of quasi-convex functions, respectively, invariance properties and we will study the quotient of analytical representations of starlikeness and convexity with respect to $(2j,k)$ - symmetric points i.e. we will study expression

$$\frac{\left[zf'(z)\right]' / f'_{2j,k}(z)}{zf'(z) / f_{2j,k}(z)} = \frac{1+zf''(z)/f'(z)}{zf'_{2j,k}(z)/f_{2j,k}(z)}$$

And obtain necessary conditions that will embed f in the class $\mathcal{S}^*_{(2j,k)}(A,B).$

To prove the main results we need the following Lemma

Lemma 1.6. [7] Let N be an regular and D starlike in $\mathcal{U}$ and $N(0)=D(0)=0.$ Then for $-1 \le B < A \le 1.$

$$\frac{N'(z)}{D'(z)} \prec \frac{1+Az}{1+Bz}$$

Implies that

$$\frac{N(z)}{D(z)} \prec \frac{1+Az}{1+Bz}.$$

Lemma 1.7. [4] If $g(z) \in \mathcal{S}^*(A,B)$, *then*

$$G(z) = \frac{m+1}{z^m}\int_0^z t^{m-1} g(t)\,dt \in \mathcal{S}^*(A,B).$$

Lemma 1.8. [2] Let Ω be a subset of the complex plan $\mathbb{C}$ and let the function $\psi : \mathbb{C}^2 * \mathcal{U} \to \mathbb{C}$ satisfy $\psi\left(Me^{i\theta}, Ne^{i\theta}; z\right) \notin \Omega$ for all real $\theta,\ N \ge M$ and for all $z \in \mathcal{U},$ if the function $p(z)$ is analytic in $\mathcal{U},\ p(0)=0$ and $\psi\left(p(z), zp'(z); z\right) \in \Omega$ for all $z \in \mathcal{U}$ then $\left|p(z)\right| < M,\ z \in \mathcal{U}.$

Lemma 1.9. Let $f \in \mathcal{A},\ k \ge 2$ is a natural number, $j = 0,1,2,\dots,\mathrm{k}-1,$ and $-1 \le B < A \le 1.$ Also, let $\Omega = \mathbb{C} \setminus \Omega_1$, where

$$\Omega_1 = \left\{ 1 + N(A-B).\frac{f_{2j,k}(z)}{zf'_{2j,k}(z)}.\frac{e^{i\theta}}{\left(1+Ae^{i\theta}\right)\left(1+Be^{i\theta}\right)} : z \in \mathcal{U},\ \theta \in \mathbb{R},\ N \ge 1 \right\}$$

If

$$\frac{\left[zf'(z)\right]' / f'_{2j,k}(z)}{zf'(z) / f_{2j,k}(z)} \in \Omega,\ z \in \mathcal{U},$$

then $f \in \mathcal{S}^*_{(2j,k)}(A,B).$

Proof: Let $p(z) = \dfrac{f_{2j,k}(z)}{zf'_{2j,k}(z)}$, so that $p(z) = \dfrac{1+Aw(z)}{1+Bw(z)}$, where w is analytic and $\left|w(z)\right| < 1$ for all $z \in \mathcal{U}.$ Since both functions are analytic in $\mathcal{U}$ and $p(0)=1$ and $w(0)=0.$ Using lemma 1.8 with

$$\psi(r,s,z) = 1 + (A-B).\frac{f_{2j,k}(z)}{zf'_{2j,k}(z)}.\frac{s}{(1+Ar)(1+Br)}$$

We have

$$\psi\left(w(z),zw'(z);z\right)=\frac{\left[zf'(z)\right]'/f'_{2j,k}(z)}{zf'(z)/f_{2j,k}(z)}, \qquad (1.9)$$

Since $\left|w(z)\right|<1,\ z\in\mathcal{U}.$

Equation (1.9) is equivalent to the following subordinates,

$$w(z)=\frac{1-P(z)}{Bp(z)-A}\prec z \ and \ \ p(z)\prec\frac{1+Az}{1+Bz},$$

Which proves that $f\in\mathcal{S}^*_{(2j,k)}(A,B).$

II. Main Results

Theorem 2.1. Let $f\in\mathcal{S}^*_{(2j,k)}(A,B),$ then $f_{2j,k}\in\mathcal{S}^*(A,B),$ end

$$\mathcal{S}^*_{(2j,k)}(A,B)\subset\mathcal{C}\subset\mathcal{S}.$$

Proof: For $f\in\mathcal{S}^*_{(2j,k)}(A,B),$ we can obtain $\dfrac{zf'(z)}{f_{2j,k}(z)}\prec\dfrac{1+Az}{1+Bz}.$ Substituting z by $\varepsilon^v z$ respectively $v=0,1,2,\ldots,k-1,$ then

$$\frac{\varepsilon^v zf'\left(\varepsilon^v z\right)}{f_{2j,k}\left(\varepsilon^v z\right)}\prec\frac{1+A\varepsilon^v z}{1+B\varepsilon^v z}\prec\frac{1+Az}{1+Bz}, \qquad (2.1)$$

Or

$$\frac{\varepsilon^{v-vj}zf'\left(\varepsilon^v z\right)}{f_{2j,k}(z)}\prec\frac{1+Az}{1+Bz}, \qquad (2.2)$$

and

$$\frac{\varepsilon^{vj-v}z\overline{f'\left(\varepsilon^v\overline{z}\right)}}{f_{2j,k}(z)}\prec\frac{1+Az}{1+Bz}, \qquad (2.3)$$

By (2.2) and (2.3), we obtain

$$\frac{\frac{z}{2}\left(\varepsilon^{v-vj}f'\left(\varepsilon^v z\right)+\varepsilon^{vj-v}\overline{f'\left(\varepsilon^v\overline{z}\right)}\right)}{f_{2j,k}(z)}\prec\frac{1+Az}{1+Bz}. \qquad (2.4)$$

Let $v=0,1,2,\ldots k-1$ in (2.4), we have

$$\frac{\frac{z}{2k}\sum_{v=0}^{k-1}\left(\varepsilon^{v-vj}f'\left(\varepsilon^v z\right)+\varepsilon^{vj-v}\overline{f'\left(\varepsilon^v\overline{z}\right)}\right)}{f_{2j,k}(z)}\prec\frac{1+Az}{1+Bz}, \qquad (2.5)$$

or

$$\frac{zf'_{2j,k}(z)}{f_{2j,k}(z)}\prec\frac{1+Az}{1+Bz}.$$

That is $f_{2j,k}(z) \in \mathcal{S}^*(A,B)$. So $f(z)$ a close-to-convex.

Corollary 2.2. Let $f \in \mathcal{K}_{2j,k}(A,B)$. Then $f_{2j,k} \in \mathcal{K}(A,B)$, end

$$\mathcal{K}_{2j,k}(A,\mathrm{B}) \subset \mathcal{C} \subset \mathcal{S}.$$

Theorem 2.3. Let $f \in \mathcal{S}^*_{(2j,k)}(A,B)$ Then so does

$$F(z) = \frac{m+1}{z^m}\int_0^z t^{m-1} f(t)\,dt \tag{2.6}$$

For $m = 1,2,3,\ldots.$

Proof: By using the equation (2.6), we have

$$F_{2j,k}(z) = \frac{m+1}{z^m}\int_0^z t^{m-1} f_{2j,k}(t)\,dt,$$

And

$$\frac{zF'(z)}{F(z)} = -m + \frac{z^m f(z)}{\int_0^z t^{m-1} f(t)\,dt}.$$

Hence

$$\frac{zF'(z)}{F_{2j,k}(z)} = \left(-m + \frac{z^m f(z)}{\int_0^z t^{m-1} f(t)\,dt}\right)\frac{F(z)}{F_{2j,k}(z)} \tag{2.7}$$

$$= \frac{z^m f(z) - m\int_0^z t^{m-1} f(t)\,dt}{\int_0^z t^{m-1} f_{2j,k}(t)\,dt} \tag{2.8}$$

$$:= \frac{N(z)}{D(z)}.$$

By Theorem 2.1 then $f_{2j,k}(z) \in \mathcal{S}^*(A,B)$, and by Lemma 1.7 we note that $F_{2j,k}(z) \in \mathcal{S}^*(A,B)$, differentiating (2.8), we have

$$\frac{N'(z)}{D'(z)} = \frac{zf'(z)}{f_{2j,k}(z)} \prec \frac{1+Az}{1+Bz},$$

Lemma 1.6 gives $F(z) \in \mathcal{S}^*_{(2j,k)}(A,B).$

By Lemma 1.9, we can prove the following Theorem.

Theorem 2.4. Let $f \in \mathcal{A},\ k \geq 2$ is a natural number, $j = 0,1,2,\ldots,k-1,$ and $-1 \leq B < A \leq 1,$ and $\mu > 1.$ Also let

$$\lambda \equiv \begin{cases} \dfrac{2\mu\sqrt{A|B|}}{1-AB}, & if\ \ AB<0\ and\ 4AB \le (A+B)(1+AB) \le 4A|B|, \\ \dfrac{(A-B)\mu}{(1+|A|)(1+|B|)}, & otherwise. \end{cases}$$

If

$$\left|\frac{zf'_{2j,k}(z)}{f_{2j,k}(z)}\right| > \frac{1}{\mu}$$

And

$$\left|\frac{[zf'(z)]'/f'_{2j,k}(z)}{zf'(z)/f_{2j,k}(z)} - 1\right| < \lambda$$

For all $z \in \mathcal{U}$ then $f \in S^*_{(2j,k)}(A,B).$

Proof: Let $\chi = \{w : |w-1| < \lambda\}$ be a subset defined in the complex plane $\mathbb{C}$, by Lemma 1.9, to prove this theorem it is enough to show that $\chi \subseteq \Omega,\ i.e.\ \chi \cap \Omega_1 = \phi$ If $w \in \Omega_1$ then for some $z \in \mathcal{U},\ \theta \in \mathbb{R}$ and $N \ge 1.$ we have

$$|w-1| = \left| N(A-B).\frac{f_{2j,k}(z)}{zf'_{2j,k}(z)}.\frac{e^{i\theta}}{(1+Ae^{i\theta})(1+Be^{i\theta})}\right|$$

$$> \frac{(A-B).\mu}{|1+Ae^{i\theta}|.|1+Be^{i\theta}|} = \frac{(A-B).\mu}{\sqrt{1+A^2+2At}.\sqrt{1+B^2+2Bt}} \equiv h(t),$$

where $t = \cos\theta \in [-1,1].$ If we show that $h(t) \ge \lambda$ for all $t \in [-1,1].$ For $0 \le B < A, AB \ge 0\ and\ h(t) \ge h(1) = \dfrac{(A-B).\mu}{(1+A).(1+B)} = \lambda\ and\ t \in [-1,1].$ Similarly, if $B < A \le 0$ then $h(t) \ge h(-1) = \dfrac{(A-B).\mu}{(1-A).(1-B)} = \lambda\ and\ t \in [-1,1].$ Finally, if $B < 0 < A,\ i.e.\ AB < 0.$ Then the function $h(t)$ attains its minimal value for $t_* = -\dfrac{(A+B)(1+AB)}{4AB},$ which is in $[-1,1]$ if and only if $4AB \le (A+B)(1+AB) \le 4A|B|.$ That value is $h(t_*) = \dfrac{2\mu\sqrt{A|B|}}{1-AB} = \lambda.$ It will imply that $w \notin \chi$ and the proof is complete.

Corollary2.5. Let $f \in \mathcal{A},\ k \ge 2$ is a natural number, $j = 0,1,2,\ldots,k-1,$ and $-1 \le B \le -3+2\sqrt{2} = -0.172\ldots$ and $\mu > 1.$ Also, let $\left|zf'_{2j,k}(z)/f_{2j,k}(z)\right| > 1/\mu$ for all $z \in \mathcal{U}.$

(i) If $0 < A \le |B|$ and

$$\left|\frac{\left[zf'(z)\right]'/f'_{2j,k}(z)}{zf'(z)/f_{2j,k}(z)} - 1\right| < \frac{2\mu\sqrt{A|B|}}{1-AB},$$

for all $z \in \mathcal{U}$ then $f \in \mathcal{S}^*_{(2j,k)}(A,B)$.

(ii) If $|B| \le A \le 1$ and

$$\left|\frac{\left[zf'(z)\right]'/zf'_{2j,k}(z)}{zf'(z)/f_{2j,k}(z)} - 1\right| < \frac{(A-B)\mu}{(1+|A|)(1+|B|)},$$

for all $z \in \mathcal{U}$ then $f \in \mathcal{S}^*_{(2j,k)}(A,B)$.

Proof: Since $AB < 0$ in (i) and (ii). So, to prove (i) it is enough to show that

$$4AB \le (A+B)(1+AB) \le 4A|B|,$$

and the rest follows Theorem 2.4. Indeed, if $-1 \le B \le -3+2\sqrt{2}$ and $0 < A \le |B|$ then $A+B \le 0,$ and the second inequality is obvious. The first inequality, $4AB \le (A+B)(1+AB),$ is equivalent to $\frac{A}{(1-A)^2} \ge -\frac{B}{(1-B)^2}$. This one is also true because $\frac{A}{(1-A)^2} \ge \frac{1}{4} \ge -\frac{B}{(1-B)^2}$ for all A and B in the specified range. Similarly we can prove (ii).

For special choice $A = 1-2\alpha\,(0 \le \alpha < 1)$ and $B = -1$ in Theorem 2.4 we can get the following corollaries.

Corollary 2.6.: Let $f \in \mathcal{A}$, $k \ge 2$ is a natural number, $j = 0,1,2,....,k-1,(0 \le \alpha < 1)$ and $\mu > 1.$ Also let

$$\lambda_1 \equiv \begin{cases} \mu/2, \ if \ 0 \le \alpha \le 1/2 \\ \dfrac{1-\alpha}{2\alpha}\mu, if \ 1/2 < \alpha < 1. \end{cases}$$

If

$$\left|\frac{zf'_{2j,k}(z)}{f_{2j,k}(z)}\right| > \frac{1}{\mu} \ and \ \left|\frac{\left[zf'(z)\right]'/f'_{2j,k}(z)}{zf'(z)/f_{2j,k}(z)} - 1\right| < \lambda_1$$

And for all $z \in U$ then $f \in \mathcal{S}^*_{2j,k}(\alpha)$.

Proof: First, let $\lambda_1 = \frac{(1-\alpha)\mu}{1+|1-2\alpha|} = \frac{(A-B)\mu}{(1+|A|)(1+|B|)}.$ Further, if $0 \le \alpha \le \frac{1}{2}$ then $A \ge 0,\ AB \ge 0$ and the conclusion of the Corollary follows since $\lambda_1 = \lambda.$ In the case when $\frac{1}{2} < \alpha < 1,$

we have $A>0$ and $AB<0$ but $(A+B)(1+AB)=-4\alpha^2>4(1-2\alpha)=4A|B|.$ Again the conclusion follows because of $\lambda_1=\lambda.$

For $\alpha=0$ in Corollary 2.6 we obtain the following corollary.

Corollary 2.7. Let $f\in\mathcal{A}, k\ge 2$ is a natural number, $j=0,1,2,...,k-1,$ and $\mu>1.$ Also, let $\left|z\,f'_{2j,k}(z)/f_{2j,k}(z)\right|>1/\mu$ for all $z\in\mathcal{U}.$ If

$$\left|\frac{[zf'(z)]'/f'_{2j,k}(z)}{zf'(z)/f_{2j,k}(z)}-1\right|<\frac{\mu}{2}$$

For all $z\in\mathcal{U}$ then $f\in\mathcal{S}^*_{(2j,k)}(0)=\mathcal{S}^*_{(2j,k)}(-1,1)=\mathcal{S}^*_{(2j,k)}.$

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